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ON THE APPLICATION OF NEWTON'S METHOD TO THE SOLUTION OF NONLINEAR BOUNDARY-VALUE PROBLEMS

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ABSTRACT: Evaluation of the applicability of several versions of Newton's method to the solution of boundary-value problems for systems of ordinary differential equations of the first and second order. The boundary value problems studied are encountered in the analysis of systems described by ordinary differential equations or when applying the method of straight lines to boundary-value problems for partial differential equations.

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The present article takes up the application of certain variants of Newton's method to the solution of boundary-value problems for systems of first- and second-order nonlinear ordinary differential equations. /17*

One may encounter boundary-value problems of this kind directly by considering objects described by ordinary differential equations or one may encounter them while applying the method of straight lines to boundary-value problems for partial differential equations.

§ 1. A boundary-value problem for second-order nonlinear differential equations with nonlinear boundary conditions

Suppose that we need to solve the boundary-value problem

$$\bar{u}'' = \bar{f}(x, \bar{u}, \bar{u}') \text{ in } D\{0 \leqslant x \leqslant l, \|\bar{u} - \bar{u}_0\|_1 < c\}, \; c > 0, \tag{1}$$

$$\bar{\varphi}(\bar{u}(0), \bar{u}'(0)) = 0, \; \bar{\psi}(\bar{u}(l), \bar{u}'(l)) = 0, \tag{2}$$

where $\bar{u}$, $\bar{f}$, $\bar{\varphi}$, $\bar{\psi}$ are p-dimensional vectors, $\bar{u}_0 = \bar{u}_0(x)$ is a continuously differentiable vector on $[0, l]$ taken as the zeroth approximation to the solution of the problem (1), (2), and the norms of the vectors and matrices in question are defined by the relations:

*Numbers in the margin indicate pagination in the foreign text.

(1) if $u = \{u_1(x), u_2(x), \ldots, u_p(x)\}$, then

$$\|\bar{u}\|_0 = \max_{0\leqslant x\leqslant l} \max_{1\leqslant i\leqslant p} |u_i(x)|, \|\bar{u}\|_1 = \|\bar{u}\|_0 + \|\bar{u}'\|_0; \tag{3}$$

(2) if

$$A = \begin{pmatrix} a_{11}(x) \ldots a_{1p}(x) \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{p1}(x) \ldots a_{pp}(x) \end{pmatrix}, \text{ then } \|A\| = \max_{0\leqslant x\leqslant l} \max_{1\leqslant i\leqslant p} \sum_{k=1}^{p} |a_{ik}(x)|. \tag{4}$$

Let us assume that the functions $\bar{f}$, $\bar{\varphi}$, and $\bar{\psi}$ satisfy the following conditions; /18

(α) $\bar{f} = \bar{f}(x, \bar{u}, \bar{u}')$ is continuous in the region D with respect to the set of arguments $x, u_1, \ldots, u_p, u_1', \ldots, u_p'$ and has continuous first and second partial derivatives with respect to $u_1, \ldots, u_p, u_1', \ldots, u_p'$. Also, for $\bar{f}$ and the Jacobian matrices

$$\bar{f}_{\bar{u}} = \left(\frac{\partial f_i}{\partial u_j}\right), \bar{f}_{\bar{u}'} = \left(\frac{\partial f_i}{\partial u_j'}\right), \bar{f}^{(i)}_{\bar{u}\bar{u}} = \left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right),$$
$$\bar{f}^{(i)}_{\bar{u}\bar{u}'} = \left(\frac{\partial^2 f_i}{\partial u_j \partial u_k'}\right), \bar{f}^{(i)}_{\bar{u}'\bar{u}'} = \left(\frac{\partial^2 f_i}{\partial u_j' \partial u_k'}\right) \tag{5}$$

the inequalities

$$\max_{D, 1\leqslant i\leqslant p} \{\|\bar{f}\|, \|\bar{f}_{\bar{u}}\|, \|\bar{f}_{\bar{u}'}\|, \|\bar{f}^{(i)}_{\bar{u}\bar{u}}\|, \|\bar{f}^{(i)}_{\bar{u}\bar{u}'}\|, \|\bar{f}^{(i)}_{\bar{u}'\bar{u}'}\|\} \leqslant m_1. \tag{6}$$

hold.

(β) $\bar{\varphi}(\bar{u}(0), \bar{u}'(0))$ and $\bar{\psi}(\bar{u}(l), \bar{u}'(l))$ are twice continuously differentiable with respect to their arguments $u_1(0), \ldots, u_p(0), u_1'(0), \ldots, u_p'(0)$ and $u_1(l), \ldots, u_p(l), u_1'(l), \ldots, u_p'(l)$ respectively. Also, for the Jacobian matrices of these vector-valued functions,

$$\max_{D, 1\leqslant i\leqslant p} \{\|\bar{\varphi}\|, \|\bar{\varphi}_{\bar{u}}\|, \|\bar{\varphi}_{\bar{u}'}\|, \|\bar{\varphi}^{(i)}_{\bar{u}\bar{u}}\|, \|\bar{\varphi}^{(i)}_{\bar{u}\bar{u}'}\|, \|\bar{\varphi}^{(i)}_{\bar{u}'\bar{u}'}\|\} \leqslant m_2, \tag{7}$$

$$\max_{D, 1\leqslant i\leqslant p} \{\|\bar{\psi}\|, \|\bar{\psi}_{\bar{u}}\|, \|\bar{\psi}_{\bar{u}'}\|, \|\bar{\psi}^{(i)}_{\bar{u}\bar{u}}\|, \|\bar{\psi}^{(i)}_{\bar{u}\bar{u}'}\|, \|\bar{\psi}^{(i)}_{\bar{u}'\bar{u}'}\|\} \leqslant m_2. \tag{8}$$

$$\gamma) \text{ Det} \begin{pmatrix} \bar{\varphi}_{\bar{u}} & \bar{\varphi}_{\bar{u}'} \\ \bar{\psi}_{\bar{u}} & l\bar{\psi}_{\bar{u}} + \bar{\psi}_{\bar{u}'} \end{pmatrix} \neq 0 \text{ for } \bar{u} = \bar{u}_0(x), \ 0 \leqslant x \leqslant l \tag{9}$$

and, consequently, the inverse matrix

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} \bar{\varphi}_{\bar{u}} & \bar{\varphi}_{\bar{u}'} \\ \bar{\psi}_{\bar{u}} & l\bar{\psi}_{\bar{u}} + \bar{\psi}_{\bar{u}'} \end{pmatrix}^{-1} \text{ and } \bar{u} = \bar{u}_0(x),\ 0 \leqslant x \leqslant l. \tag{10}$$

exists.

(δ) The linear boundary-value problem for $\bar{U}(x)$

$$\bar{U}'' - \bar{f}^0_{\bar{u}}\bar{U} - \bar{f}^0_{\bar{u}'}\bar{U}' = \bar{\omega}_0(x),\ 0 < x < l, \tag{11}$$

$$\bar{\varphi}^0_{\bar{u}}\bar{U}(0) + \bar{\varphi}^0_{\bar{u}'}\bar{U}'(0) = \bar{\omega}_1, \tag{12}$$

$$\bar{\psi}^0_{\bar{u}}\bar{U}(l) + \bar{\psi}^0_{\bar{u}'}\bar{U}'(l) = \bar{\omega}_2, \tag{13}$$

where

$$\left.\begin{aligned} &\bar{f}^0 = \bar{f}(x, \bar{u}_0(x), \bar{u}_0'(x)), \bar{f}^0_{\bar{u}} = \bar{f}_{\bar{u}}(x, \bar{u}_0(x), \bar{u}_0'(x)), \bar{f}^0_{\bar{u}'} = \bar{f}^0_{\bar{u}'}(x, \bar{u}_0(x), \bar{u}_0'(x)) \\ &\bar{\varphi}^0 = \bar{\varphi}(\bar{u}_0(0), \bar{u}_0'(0)), \bar{\varphi}^0_{\bar{u}} = \bar{\varphi}_{\bar{u}}(\bar{u}_0(0), \bar{u}_0'(0)), \bar{\psi}^0 = \bar{\psi}(\bar{u}_0(l), \bar{u}_0'(l)), \dots \end{aligned}\right\} \tag{13$_1$}$$

has a unique solution for arbitrary $\bar{\omega}_0(x)$, $\bar{\omega}_1$, $\bar{\omega}_2$, where $\bar{\omega}_0(x)$ is continuous on $[0, l]$, and the Green's matrix function /19

$$G(x, \xi) = \begin{pmatrix} g_{11}(x, \xi) \dots g_{1p}(x, \xi) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ g_{p1}(x, \xi) \dots g_{pp}(x, \xi) \end{pmatrix}, \text{ defined by the conditions:}$$

$$\left.\begin{aligned} &G''(x, \xi) - \bar{f}^0_{\bar{u}}G(x, \xi) - \bar{f}^0_{\bar{u}'}G'(x, \xi) = E\delta(x - \xi), \\ &\bar{\varphi}^0_{\bar{u}}G(0, \xi) + \bar{\varphi}^0_{\bar{u}'}G'(0, \xi) = 0, \\ &\bar{\psi}^0_{\bar{u}}G(l, \xi) + \bar{\psi}^0_{\bar{u}'}G'(l, \xi) = 0, \end{aligned}\right\} \tag{14}$$

exists, where $\delta(x - \xi)$ is Dirac's delta function and E is the pth-order unit square matrix. The solution of the boundary-value problem (13) for $\omega_1 = \omega_2 = 0$ can be written in the form

$$\bar{U}(x) = \int_0^l G(x, \xi)\bar{\omega}_0(\xi)\,d\xi. \tag{15}$$

We shall assume that

$$\|G\| \leqslant M_1 \text{ и } \|G_x'\| \leqslant M_2 \text{ for } 0 \leqslant x, \xi \leqslant l. \tag{16}$$

1°. The first variant of Newton's method. Beginning with the zeroth approximation $\bar{u}_0(x)$ (cf. equation (1)), we shall seek all subsequent approximations

by solving successively, for n = 0, 1, 2, ..., the linear boundary-value problems

$$\bar{u}'_{n+1} - \bar{f}^n_{\bar{u}} \bar{u}_{n+1} - \bar{f}^n_{\bar{u}'} \bar{u}'_{n+1} = \bar{f}^n - \bar{f}^n_{\bar{u}} \bar{u}_n - \bar{f}^n_{\bar{u}'} \bar{u}'_n, \quad 0 \leqslant x \leqslant l, \tag{17}$$

$$\bar{\varphi}^n_{\bar{u}} \bar{u}_{n+1}(0) + \bar{\varphi}^n_{\bar{u}'} \bar{u}'_{n+1}(0) = \bar{\varphi}^n_{\bar{u}} \bar{u}_n(0) + \bar{\varphi}^n_{\bar{u}'} \bar{u}'_n(0) - \bar{\varphi}^n, \tag{18}$$

$$\bar{\psi}^n_{\bar{u}} \bar{u}_{n+1}(l) + \bar{\psi}^n_{\bar{u}'} \bar{u}'_{n+1}(l) = \bar{\psi}^n_{\bar{u}} \bar{u}_n(l) + \bar{\psi}^n_{\bar{u}'} \bar{u}_n(l) - \bar{\psi}^n, \tag{19}$$

where

$$\bar{f}^n = \bar{f}(x, \bar{u}_n, \bar{u}'_n), \ \bar{f}^n_{\bar{u}} = \bar{f}_{\bar{u}}(x, \bar{u}_n, \bar{u}'_n), \ \bar{f}^n_{\bar{u}'} = \bar{f}_{\bar{u}'}(x, \bar{u}_n, \bar{u}'_n),$$
$$\bar{\varphi}^n = \bar{\varphi}(\bar{u}_n(0), \bar{u}'_n(0)), \ \bar{\varphi}^n_{\bar{u}} = \bar{\varphi}(\bar{u}_n(0), \bar{u}'_n(0)), \ \bar{\psi}^n = \bar{\psi}(\bar{u}_n(l), \bar{u}'_n(l)), \ldots \tag{20}$$

The linear boundary-value problem (17) - (19) can in fact be solved by reducing it to a boundary-value problem for a system of first-order linear equations and employing some sort of sweep procedure, for example, that described by Godunov [3].

Theorem 1. If the norm of the difference between the first approximation $\bar{u}_1(x)$ and the zeroth approximation $\bar{u}_0(x)$ satisfies the inequality

$$\|\bar{u}_1 - \bar{u}_0\| \leqslant \eta_0,$$

where η_0 is a sufficiently small number, then, if conditions (α) - (δ) are satisfied, all the approximations $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n, \ldots$ lie in the region D and constitute a fundamental sequence in the norm $\|\|_1$. Also, the limit of this sequence $\bar{u}^*(x)$ is the solution of the boundary-value problem (1), (2), and the rate of convergence is characterized by the inequality /20

$$\|\bar{u}_n - \bar{u}^*\|_1 \leqslant \frac{\eta_0}{2^{n-1}} (2h_0)^{2^n - 1}, \tag{21}$$

where

$$0 < h_0 < \frac{1}{2}.$$

Proof: We shall prove Theorem 1 by showing that the conditions of a familiar theorem [1] are satisfied.

Theorem K. Suppose that the operator P(x) = y that maps a region D_X contained in a Banach space X into a region $D_{P(x)}$ contained in a Banach space Y satisfies the following conditions:

(1) The operator P(x) is twice differentiable in the Fréchet sense in D_X and

$$\|P''(x)\|_{D_x} \leqslant K = \text{const} < +\infty, \tag{22}$$

(2) a bounded linear operator $\Gamma_0 = P'(x_0)$ mapping X into Y has an inverse $\Gamma_0^{-1} = [P'(x_0)]^{-1}$. Also, the norm of that inverse operator has a bound

$$\|\Gamma_0^{-1}\| \leqslant B_0 \tag{23}$$

and satisfies the inequality

$$\|x_1 - x_0\| = \|\Gamma_0^{-1} P(x_0)\| \leqslant \eta_0, \tag{24}$$

(3) and the inequality

$$h_0 = B_0 K \eta_0 < \frac{1}{2}, \tag{25}$$

is satisfied.

(4) the ball $Ш(x_0, r_0)$ of radius $r_0 = N(h_0)\eta_0 = \frac{(1-\sqrt{1-2h_0})\eta_0}{h_0}$ is contained in the region D_x.

Then, it is possible to define a sequence $\{x_n\}$ by

$$x_{n+1} = x_n - \Gamma_n^{-1} P(x_n), \quad n = 0, 1, 2, \ldots, \tag{26}$$

Each x_n lies in the ball $Ш(x_0, r_0)$, and the sequence converges to a solution x^* of the equation

$$P(x) = 0 \tag{27}$$

at a rate characterized by the inequality

$$\|x_n - x^*\| \leqslant \frac{\eta_0}{2^{n-1}} (2h_0)^{2^n - 1}. \tag{28}$$

To reduce the proof of Theorem 1 to a verification of the conditions of Theorem K, let us look at the operator /21

$$P(\bar{u}) = \begin{cases} \bar{g}(\bar{u}, \bar{u}', \bar{u}'') = \bar{u}'' - \bar{f}(x, \bar{u}, \bar{u}'), \\ \bar{\varphi}(\bar{u}(0), \bar{u}'(0)), \\ \bar{\psi}(\bar{u}(l), \bar{u}'(l)), \end{cases} \tag{29}$$

that is defined in a region $D: \{0 \leqslant x \leqslant l, \ \|\bar{u} - \bar{u}_0\|_1 < c\}$ contained in the Banach space $X = C_2[0, l; p]$ of twice continuously differentiable p-dimensional vectors $\bar{u}(x) = \{u_1(x), u_2(x), \ldots, u_p(x)\}$ on the interval $[0, l]$ and that assigns to each vector $\bar{u}(x) \in D$ the 3 p-dimensional vector

$$P(\bar{u}) = \{\bar{g}(x, \bar{u}(x), \bar{u}'(x)), \bar{\varphi}(\bar{u}(0), \bar{u}'(0)), \bar{\psi}(\bar{u}(l), \bar{u}'(l))\} = \{\bar{g}, \bar{\varphi}, \bar{\psi}\} \tag{30}$$

Translators note: The symbol Ш is the Cyrillic letter, corresponding to sh, with which the Russian word "shar", meaning "ball", begins.

in the Banach space $Y=C_0[0,\ l;\ p]\times R^p\times R^p$ which is the metric product of the space of p-dimensional continuous vectors $C_0[0,\ l;\ p]$ on $[0,\ l]$ and the p-dimensional Euclidean space R^p (taken twice).

We shall use the above-defined norms of the vectors in X and the norms of the matrices. We define the norm in Y as the sum of the norms in $C_0[0,\ l;\ p]$ R^p, and R^p:

$$\|P(\bar{u})\|_Y = \|\bar{g}\|_0 + \|\bar{\varphi}\|_0 + \|\bar{\psi}\|_0.$$

We rewrite the algorithm (26) of Newton's method for our case in the form

$$P(\bar{u}_n) + P'(\bar{u}_n)(\bar{u}_{n+1} - \bar{u}_n) = 0. \tag{26*}$$

In accordance with (29), equation (26*) is equivalent to the boundary-value problem

$$\bar{u}_n' - \bar{f}^n + (\bar{u}_{n+1}' - \bar{u}_n') - \bar{f}^n_{\bar{u}}(\bar{u}_{n+1} - \bar{u}_n) - \bar{f}^n_{\bar{u}'}(\bar{u}_{n+1}' - \bar{u}_n') = 0, \tag{31}$$

$$\bar{\varphi}^n + \bar{\varphi}^n_{\bar{u}}(\bar{u}_{n+1}(0) - \bar{u}_n(0)) + \bar{\varphi}^n_{\bar{u}'}(\bar{u}_{n+1}'(0) - \bar{u}_n'(0)) = 0, \tag{32}$$

$$\bar{\psi}^n + \bar{\psi}^n_{\bar{u}}(\bar{u}_{n+1}(l) - \bar{u}_n(l)) + \bar{\psi}^n_{\bar{u}'}(\bar{u}_{n+1}'(l) - \bar{u}_n'(l)) = 0. \tag{33}$$

Let us verify that condition (1) of Theorem K is satisfied. By virtue of the existence of continuous first and second partial derivatives of $\bar{f}$, $\bar{\varphi}$, and $\bar{\psi}$ that are bounded with respect to the norm (see conditions (α) and (β)) the second Fréchet differential, that is, a bilinear operator $P''(\bar{u})(\bar{v},\ \bar{w})$, mapping X^2 into Y, will exist and its norm is equal to

$$\begin{aligned}\|P''(\bar{u})\| = \sup_{\|\bar{v}\|_1\leqslant 1}\|P''(\bar{u})(\bar{v},\ \bar{v})\|_Y = \sup_{\|\bar{v}\|_1\leqslant 1}\{\|\bar{f}_{\bar{u}\bar{u}}(\bar{u})(\bar{v},\ \bar{v}) + \\ + 2\bar{f}_{\bar{u}\bar{u}'}(\bar{v},\ \bar{v}') + \bar{f}_{\bar{u}'\bar{u}'}(\bar{v}',\ \bar{v}')\|_0 + \|\bar{\varphi}_{\bar{u}\bar{u}}(\bar{u}(0),\ \bar{u}'(0)),\ (\bar{v}(0),\ \bar{v}(0)) + \\ + 2\bar{\varphi}_{\bar{u}\bar{u}'}(\bar{u}(0),\ \bar{u}'(0))(\bar{v}(0),\ \bar{v}'(0)) + \bar{\varphi}_{\bar{u}'\bar{u}'}(\bar{u}'(0),\ \bar{u}'(0)(\bar{v}'(0),\ \bar{v}'(0))\| + \\ + \|\bar{\psi}_{\bar{u}\bar{u}}(\bar{u}(l),\ \bar{u}'(l))(\bar{v}(l),\ \bar{v}(l)) + \\ + 2\bar{\psi}_{\bar{u}\bar{u}'}(\bar{u}(l),\ \bar{u}'(l))(\bar{v}(l),\ \bar{v}'(l)) + \bar{\psi}_{\bar{u}'\bar{u}'}(\bar{u}(l),\ \bar{u}'(l))(\bar{v}(l),\ \bar{v}'(l))\| \leqslant K\end{aligned} \tag{34}$$

/22

in the region $D_x=\{0\leqslant x\leqslant l,\ \|\bar{u}-\bar{u}\|_1<c\}$. Thus, condition (1) of Theorem K is satisfied.

Let us verify that condition (2) of Theorem K is satisfied. The existence of the operator $\Lambda_0^{-1} = [P'(\bar{u}_0)]^{-1}$, is equivalent to a uniquely solvable linear boundary-value problem (13) for arbitrary right-hand members $\omega_0(x)$, ω_1, ω_2, where $\bar{\omega}_0(x)$ is an arbitrary p-dimensional vector that is continuous on $[0,\ l]$ and $\bar{\omega}_1$ and $\bar{\omega}_2$ are arbitrary constant p-dimensional vectors. This situation obtains by virtue of assumption (δ). Let us now show that the operator $\Lambda_0^{-1} = [P'(\bar{u}_0)]^{-1}$, mapping the vector $(\bar{\omega}_0(x),\ \bar{\omega}_1,\ \bar{\omega}_2)\in Y=C_0[0,\ l;\ p]\times R^p\times R^p$ into

X is bounded. To do this, let us use Green's function (see condition (δ)) and let us derive a suitable representation for the solution of the problem (13). Let us seek a solution of (13) in the form

$$\bar{U}(x) = \bar{V}(x) + \bar{W}(x), \tag{35}$$

where

$$\bar{W}(x) = C_1 + xC_2 \tag{36}$$

is a linear function of x with constant p-dimensional vector-valued coefficients C_1 and C_2 that satisfies the nonhomogeneous boundary conditions of problem (13) and $\bar{V}(x)$ is a solution of the following linear boundary-value problem with homogeneous boundary conditions:

$$\begin{gathered}\bar{V}'' - \bar{f}_u^0 \bar{V} - \bar{f}_{u'}^0 \bar{V}' = \bar{\omega}_0(x) + \bar{f}_u^0 (C_1 + xC_2) + \bar{f}_{u'}^0 C_2, \\ \bar{\varphi}_u^0 \bar{V}(0) + \bar{\varphi}_{u'}^0 \bar{V}'(0) = 0, \\ \bar{\psi}_u^0 \bar{V}(l) + \bar{\psi}_{u'}^0 \bar{V}'(l) = 0.\end{gathered} \tag{13*}$$

By virtue of condition (γ), the solution of the system of linear algebraic equations obtained when we substitute $\bar{W}(x)$ into the boundary conditions of problem (13)

$$\begin{gathered}\bar{\varphi}_u^0 C_1 + \bar{\varphi}_{u'}^0 C_2 = \bar{\omega}_1, \\ \bar{\psi}_u^0 C_1 + (l\bar{\psi}_u^0 + \bar{\psi}_{u'}^0) C_2 = \bar{\omega}_2,\end{gathered} \tag{37}$$

can be represented in the form

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}, \tag{38}$$

and, consequently

$$C_1 = T_1\bar{\omega}_1 + T_2\bar{\omega}_2, \quad C_2 = T_3\bar{\omega}_1 + T_4\bar{\omega}_2. \tag{39}$$ /23

Substituting these expressions into (13*) and (36), representing the solution of the problem (13*) with the aid of Green's function, and combining it with (36), we obtain

$$\begin{aligned}\bar{U}(x) = \int_0^l G(x, \xi) \{\bar{\omega}_0(\xi) + \bar{f}_u^0(\xi) [(T_1\bar{\omega}_1 + T_2\bar{\omega}_2) + \xi(T_3\bar{\omega}_1 + T_4\bar{\omega}_2)] + \\ + \bar{f}_{u'}^0(\xi)(T_3\bar{\omega}_1 + T_4\bar{\omega}_2)\} d\xi + (T_1\bar{\omega}_1 + T_2\bar{\omega}_2) + x(T_3\bar{\omega}_1 + T_4\bar{\omega}_2).\end{aligned} \tag{40}$$

Differentiating equation (40) with respect to x, we find

$$\bar{U}'(x) = -\int_0^l G_x'(x, \xi)\{\omega_0(\xi) + \bar{f}_{\bar{u}}^0(\xi)[(T_1\bar{\omega}_1 + T_2\bar{\omega}_2) + \xi(T_3\bar{\omega}_1 + T_4\bar{\omega}_2)] +$$

$$+ \bar{f}_{\bar{u}'}^0(\xi)(T_3\bar{\omega}_1 + T_4\bar{\omega}_2)\}d\xi + (T_3\bar{\omega}_1 + T_4\bar{\omega}_2),$$

$$\|\bar{U}\|_1 \leqslant B_0\{\|\bar{\omega}_0\|_0 + \|\bar{\omega}_1\|_0 + \|\bar{\omega}_2\|\}, \tag{41}$$

where the constant B_0 is expressed in terms of m_1, M_1, M_2, $\| T_i \|$ for $i = 1, 2, 3, 4$ and it depends on l. But the relation (41) means that

$$\|\Lambda_0^{-1}\| = \|[P'(\bar{u}_0)]^{-1}\| \leqslant B_0. \tag{23}$$

Thus, inequality (23) of condition 2 of Theorem K is satisfied. Let us assume, finally, that the number η_0 in the inequality

$$\|\bar{u}_1 - \bar{u}_0\|_1 \leqslant \eta_0 \tag{42}$$

is sufficiently small that

$$h_0 = B_0 K \eta_0 < \frac{1}{2} \tag{25}$$

and that the ball $\mathfrak{U}(\bar{u}_0, \bar{r}_0): \{0 \leqslant x \leqslant l, \|\bar{u} - \bar{u}_0\| \leqslant r_0\}$ is contained in the region $D: \{0 \leqslant x \leqslant l, \|\bar{u} - \bar{u}_0\| < c\}$ if

$$r_0 = N(h_0)\eta_0 = \frac{1 - \sqrt{1 - 2h_0}}{h_0}\eta_0.$$

Then, all conditions of Theorm K are satisfied. This completes the proof of Theorem 1.

Remark: If we assume that the region D is contained in the ball

$$\mathfrak{U}(\bar{u}_0, r_1): \{0 \leqslant x \leqslant l, \|\bar{u} - \bar{u}_0\| \leqslant r_1\},$$

where

$$r_1 = \tilde{N}(h_0)\eta_0 = \frac{1 + \sqrt{1 - 2h_0}}{h_0}\eta_0,$$

we see, in accordance with the uniqueness theorem of [1] that the solution of the nonlinear problem (1), (2) is unique in this region.

2°. The second variant of Newton's method (the "modified" Newton's method). Beginning with the zeroth approximation $\bar{u}_0$ (see equation (1)), we can seek all subsequent approximations in accordance with the formulas /24

$$\bar{u}'_{n+1} - \bar{f}^0_{\bar{u}}\,\bar{u}_{n+1} - \bar{f}^0_{\bar{u}'}\,\bar{u}'_{n+1} = \bar{f}^n - \bar{f}^0_{\bar{u}}\,\bar{u}_n - \bar{f}^0_{\bar{u}'}\,\bar{u}'_n,\ 0 \leqslant x \leqslant l, \tag{43}$$

$$\bar{\varphi}^0_{\bar{u}}\,\bar{u}_{n+1}(0) + \bar{\varphi}^0_{\bar{u}'}\,\bar{u}'_{n+1}(0) = \bar{\varphi}^0_{\bar{u}}\,\bar{u}_n(0) + \bar{\varphi}^0_{\bar{u}'}\,\bar{u}'_n(0) - \bar{\varphi}^n, \tag{44}$$

$$\bar{\psi}^0_{\bar{u}}\,\bar{u}_{n+1}(l) + \bar{\psi}^0_{\bar{u}'}\,\bar{u}'_{n+1}(l) = \bar{\psi}^0_{\bar{u}}\,\bar{u}_n(l) + \bar{\psi}^0_{\bar{u}'}\,\bar{u}'_n(l) - \bar{\psi}^n, \tag{45}$$

where the notation is as in (20). The linear boundary-value problem (43) - (45), for n = 0, 1, 2, ..., can be solved in the case of second-order equations, for example, in an obvious manner by reducing it to linear boundary-value problems for first-order equations and then applying some variant or other of the sweep method [3].

The rate of convergence of this variant of Newton's method is characterized by the inequality

$$\|\bar{u}_n - \bar{u}^*\|_1 \leqslant \frac{\eta_0}{h_0}(1 - \sqrt{1 - 2h_0})^{n+1},\ n = 0, 1, \ldots,$$

as can be proved in a manner analogous to the corresponding proof in the case of the first variant.

3°. The third variant of Newton's method (with "modified" boundary conditions). Beginning with the zeroth approximation $\bar{u}_0(x)$ (see equation (1)), we can seek all successive approximations in accordance with the formulas

$$\bar{u}'_{n+1} - \bar{f}^n_{\bar{u}}\,\bar{u}_{n+1} - \bar{f}^n_{\bar{u}'}\,\bar{u}'_{n+1} = \bar{f}^n - \bar{f}^n_{\bar{u}}\,\bar{u}_n - \bar{f}^n_{\bar{u}'}\,\bar{u}'_n,\ 0 \leqslant x \leqslant l, \tag{46}$$

$$\bar{\varphi}^0_{\bar{u}}\,\bar{u}_{n+1}(0) + \bar{\varphi}^0_{\bar{u}'}\,\bar{u}'_{n+1}(0) = \bar{\varphi}^0_{\bar{u}}\,\bar{u}_n(0) + \bar{\varphi}^0_{\bar{u}'}\,\bar{u}'_n(0) - \bar{\varphi}^n, \tag{47}$$

$$\bar{\psi}^0_{\bar{u}}\,\bar{u}_{n+1}(l) + \bar{\psi}^0_{\bar{u}'}\,\bar{u}'_{n+1}(l) = \bar{\psi}^0_{\bar{u}}\,\bar{u}_n(l) + \bar{\psi}^0_{\bar{u}'}\,\bar{u}'_n(l) - \bar{\psi}^n, \tag{48}$$

where the notation is as in (20). We can show, though no longer a reduction to a theorem of the type K but by satisfying directly some rather tedious inequalities, that the successive approximations $\bar{u}_n(x)$ obtained from formulas (46) - (48) converge to the exact solution $\bar{u}^*(x)$ of the problem (1), (2) at a rate characterized by the inequality

$$\|\bar{u}_n - \bar{u}^*\|_1 \leqslant Cq^{2^n - 1},\ \text{where } 0 < q < 1,$$

which is analogous to inequality (21) of [4].

4°. The finite-difference analogues of Newton's method. Together with the differential variants of Newton's method that we considered in subsections 1°, 2°, and 3°, we can consider their finite-difference analogues. Replacing the differential boundary-value problem (1), (2), for example with the following approximating finite-difference boundary-value problem

$$\bar{y}_{\bar{x}\bar{x}} = \bar{f}(x, \bar{y}, \bar{y}^\circ_x), \tag{49}$$

$$\bar{\varphi}(\bar{y}_0, \bar{y}_{x0}) = 0,\quad \bar{\psi}(\bar{y}_N, \bar{y}_{x,N}) = 0, \tag{50}$$

/25

where

$$\bar{y} = \bar{y}_i = \bar{y}(x_i),\ x = x_i = ih,\ i = 0, 1, \ldots, N+1,\ (N+1)h = l,$$

$$\bar{y}_{\mathring{x}} = \frac{\bar{y}_{i+1} - \bar{y}_{i-1}}{2h},\quad \bar{y}_{x\bar{x}} = \frac{\bar{y}_{i+1} - 2\bar{y}_i + \bar{y}_{i-1}}{h^2},$$

$$\bar{y}_{x0} = \frac{y_1 - y_0}{h},\quad \bar{y}_{xN} = \frac{\bar{y}_{N+1} - \bar{y}_N}{h},$$

we can apply the finite-difference analogues of the first, second, and third variants of Newton's method to the nonlinear finite-difference boundary-value problem (49), (50). The analogue of the first variant of Newton's method is defined by the iterational relationships

$$\bar{y}^{n+1}_{x\bar{x}} - \bar{f}^n_{\bar{u}}\,\bar{y}^{n+1} - \bar{f}^n_{\bar{u}'}\,\bar{y}^{n+1}_{\mathring{x}} = \bar{f}^n - \bar{f}^n_{\bar{u}}\,\bar{y}^n - \bar{f}^n_{\bar{u}'}\,\bar{y}^n_{\mathring{x}}, \tag{51}$$

$$\bar{\varphi}^n_{\bar{u}}\,\bar{y}^{n+1}_0 + \bar{\varphi}^n_{\bar{u}'}\,\bar{y}^{n+1}_{x0} = \bar{\varphi}^n_{\bar{u}}\,\bar{y}^n_0 + \bar{\varphi}^n_{\bar{u}'}\,\bar{y}^n_{x0} - \bar{\varphi}^n, \tag{52}$$

$$\bar{\psi}^n_{\bar{u}}\,\bar{y}^{n+1}_{N+1} + \bar{\psi}^n_{\bar{u}'}\,\bar{y}^{n+1}_{xN} = \bar{\psi}^n_{\bar{u}}\,\bar{y}^n_{N+1} + \bar{\psi}^n_{\bar{u}'}\,\bar{y}^n_{xN} - \psi^n, \tag{53}$$

where

$$\bar{f}^n = \bar{f}(x, \bar{y}^n, \bar{y}^n_{\mathring{x}}),\ \bar{f}^n_{\bar{u}} = \bar{f}_{\bar{u}}(x, \bar{y}^n, \bar{y}^n_{\mathring{x}}), \ldots, \bar{\psi}^n_{\bar{u}'} = \bar{\psi}_{\bar{u}'}(\bar{y}^n_N, \bar{y}^n_{xN}).$$

Solving the linear boundary-value problems (51) - (53) for n = 0, 1, 2, ..., we obtain a sequence of solutions that converges with respect to the norm

$$\|\bar{y}\|_1 = \|\bar{y}\|_0 + \|\bar{y}^{\circ}_{\mathring{x}}\|_0, \tag{54}$$

where

$$\|\bar{y}\|_0 = \max_{0 \leqslant i \leqslant N+1} \|\bar{y}_i\|_0,\ \|\bar{y}^{\circ}_{\mathring{x}}\|_0 = \max_{1 \leqslant i \leqslant N} \|\bar{y}^{\circ}_{\mathring{x}i}\|,$$

to a solution $\bar{y}$ of the non-linear finite-difference boundary-value problem (49), (50) at a rate characterized by an inequality of the form (21).

Proof of the convergence of this analogue of the first variant of Newton's method is done in a manner analogous to the proof of Theorem 1, replacing the Green's function G(x, ξ) with the corresponding Green's difference function.

We define and use the finite-difference analogues of the second and third variants of Newton's method analogously. Here, it should be noted that we can examine difference schemes with a more accurate approximation of the boundary conditions than (50) and we can consider difference schemes on a nonuniform grid.

§ 2. A Boundary-Value Problem for Nonlinear First-Order Differential Equations with Nonlinear Boundary Conditions /26

Suppose that we need to solve the boundary-value problem

$$\bar{u}' = \bar{f}(x, \bar{u}) \text{ in } D: \{0 \leqslant x \leqslant l, \ \|\bar{u} - \bar{u}_0\|_0 < c\}, \tag{1}$$

$$\bar{\varphi}(\bar{u}(0)) = 0, \quad \bar{\psi}(\bar{u}(l)) = 0, \tag{2}$$

where $\bar{u}, \bar{f}, \bar{\varphi}, \bar{\psi}$ are p-dimensional vectors, $\bar{u}_0 = \bar{u}_0(x)$ is a continuous vector on $[0, l]$ representing the "zeroth" approximation to the solution of the problem (1), (2), and the norms of the vectors and matrices are defined as in section 1. We shall assume that the functions $\bar{f}$, $\bar{\varphi}$, and $\bar{\psi}$ satisfy conditions (α) and (β) listed in section 1 except that they are independent of $u_1', \ldots, u_p'$, $u_1'(0), \ldots, u_p'(0)$, $u_1'(l), \ldots, u_p'(l)$, and the following conditions are satisfied:

(γ *, The linear boundary-value problem

$$\bar{U}' - \bar{f}_u^0 \bar{U} = \bar{\omega}_0(x), \ 0 < x < l, \tag{3}$$

$$\bar{\varphi}_u^0 \bar{U}(0) = \bar{\omega}_1, \ \bar{\psi}_u^0 \bar{U}(l) = \bar{\omega}_2, \tag{4}$$

where

$$\bar{f}_u^0 = \bar{f}_u(x, \bar{u}_0(x)), \ \bar{\varphi}_u^0 = \bar{\varphi}_u(\bar{u}_0(0)), \ \bar{\psi}_u^0 = \bar{\psi}_u(\bar{u}_0(l)),$$

has a unique solution for arbitrary vectors $\bar{\omega}_0(x)$, $\bar{\omega}_1$, $\bar{\omega}_2$, where $\bar{\omega}_0(x)$ is a continuous vector on $[0, l]$ and $\bar{\omega}_1$ and $\bar{\omega}_2$ are constant vectors. Also, we assume that the Green's matrix function of the problem (3), (4) defined by the conditions

$$G_x'(x, \xi) = \bar{f}_u^0(x) G(x, \xi) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant \xi, \ \xi \leqslant x \leqslant l, \tag{5}$$

$$\bar{\varphi}_u^0 G(0, \xi) + \bar{\psi}_u^0 G(l, \xi) = 0, \tag{6}$$

$$G(\xi + 0, \xi) - G(\xi - 0, \xi) = E. \tag{7}$$

exists:

(δ *)

$$\operatorname{Det}(\bar{\varphi}_u^0 + \bar{\psi}_u^0 \overset{l}{\underset{0}{\Omega}}(\bar{f}_u^0)) \neq 0, \tag{8}$$

and, consequently, the inverse matrix

$$T = (\bar{\varphi}^{\cdot} \quad \ldots \quad \ldots \quad \ldots)^{-1}, \tag{9}$$

exists. Here, $\overset{x}{\underset{0}{\Omega}}(\bar{f}_{\bar{u}}^{0})$ denotes the matrizant of the system (3), that is, the solution of the homogeneous equation

$$\frac{d}{dx}\left[\overset{x}{\underset{0}{\Omega}}(\bar{f}_{\bar{u}}^{0})\right]-\bar{f}_{\bar{u}}^{0}\left[\overset{x}{\underset{0}{\Omega}}(\bar{f}_{\bar{u}}^{0})\right]=0,\ 0\leqslant x\leqslant l, \tag{10}$$

that satisfies the initial condition /27

$$\overset{x}{\underset{0}{\Omega}}(\bar{f}_{\bar{u}}^{0})|_{x=0}=E, \tag{11}$$

where E is the unit matrix.

When these conditions are satisfied, the solution of the boundary-value problem (3), (4) can, for arbitrary $\bar{\omega}_0(x)$, ω_1, ω_2 (where $\bar{\omega}_0(x)$ is a continuous vector on $[0, l]$), be represented in the form

$$\bar{U}(x)=\int_0^l G(x,\xi)\,\bar{\omega}_0(\xi)\,d\xi+\overset{x}{\underset{0}{\Omega}}(\bar{f}_{\bar{u}}^{0})T(\bar{\omega}_1+\bar{\omega}_2). \tag{12}$$

We shall assume that

$$\|G\|\leqslant M_1 \text{ and } \|G_x'\|\leqslant M_2 \text{ when } 0\leqslant x,\xi\leqslant l. \tag{13}$$

1°. The first variant of Newton's method. Beginning with the zeroth approximation $\bar{u}_0(x)$ (see equation (1)), let us seek all subsequent approximations, solving the linear boundary-value problems for n = 0, 1, 2, ...:

$$\bar{u}_{n+1}'=\bar{f}^{n}+\bar{f}_{\bar{u}}^{n}(\bar{u}_{n+1}-\bar{u}_n),\ 0\leqslant x<l, \tag{14}$$

$$\bar{\varphi}^{n}+\bar{\varphi}_{\bar{u}}^{n}(\bar{u}^{n+1}(0)-\bar{u}^{n}(0))=0, \tag{15}$$

$$\bar{\psi}^{n}+\bar{\psi}_{\bar{u}}^{n}(\bar{u}^{n+1}(l)-\bar{u}^{n}(l))=0, \tag{16}$$

where

$$\bar{f}^{n}=\bar{f}(x,\bar{u}_n),\ \bar{f}_{\bar{u}}^{n}=\bar{f}_{\bar{u}}(x,\bar{u}_n),\ \bar{\varphi}^{n}=\bar{\varphi}(u_n(0)),\ \bar{\psi}^{n}=\bar{\psi}(\bar{u}_n(l)). \tag{17}$$

The linear boundary-value problem (14) - (16) can be solved for n = 0, 1, 2, 3, ..., for example, by means of some sort of sweep procedure (see, for example [3]).

Theorem 2. Suppose that the norm of the difference between the first approximation $\bar{u}_1(x)$ and the zeroth approximation $\bar{u}_0(x)$ satisfies the inequality

$$\|\bar{u}_1-\bar{u}_0\|<\eta_0, \tag{17*}$$

where η_0 is a sufficiently small positive number. Then, if conditions (α), (β), (γ^*), and (δ^*) are satisfied, the approximations $\bar{u}_1$, $\bar{u}_2$, ..., $\bar{u}_n$ will all lie in the region D and they will constitute a fundamental sequence with respect to the norm $\|\ \|_0$. Also, the limit of this sequence $\bar{u}^*(x)$ is a solution of the boundary-value problem (1), (2), and the rate of convergence is characterized by the inequality

$$\|\bar{u}_n - \bar{u}^*\|_0 \leqslant \frac{\eta_0}{2^{n-1}}(2h_0)^{2^{n-1}}, \tag{18}$$

where $0 < h_0 < 1/2$.

Proof: We shall prove Theorem 2 by showing that the conditions of Theorem K (see section 1, subsection 1°) are satisfied. Consider the operator /28

$$\bar{P}(\bar{u}) = \begin{cases} \bar{g}(x, \bar{u}, \bar{u}') = \bar{u}' - \bar{f}(x, \bar{u}), \\ \bar{\varphi}(\bar{u}(0)), \\ \bar{\psi}(\bar{u}(l)), \end{cases} \tag{19}$$

or, in other words,

$$P(\bar{u}) = (\bar{g}(x, \bar{u}, \bar{u}'),\ \bar{\varphi}(\bar{u}(0)),\ \bar{\psi}(\bar{u}(l))) = (\bar{g}, \bar{\varphi}, \psi), \tag{20}$$

that is defined in a region D: $\{0 \leqslant x \leqslant l,\ \|\bar{u} - \bar{u}_0\| < c\}$ contained in the Banach space $X = C_1[0, l;\ p]$ of continuously differentiable p-dimensional vectors $\bar{u} = \{u_1(x), ..., u_p(x)\}$ on the interval $[0, l]$ and that assigns to every vector $\bar{u}(x) \in D$ the 3p-dimensional vector (20) in the Banach space $Y = C_0[0, l;\ p] \times R^p \times R^p$ which is the metric product of the space of continuous p-dimensional vectors $C_0[0, l;\ p]$ on $[0, l]$ and the p-dimensional Euclidean space R^p (again taken twice).

Let us use the above-defined norms of the vectors in X and the matrices and vectors in Y. We rewrite the algorithm for Newton's method for this case in the form (see equations (26) and (26*) of section 1):

$$\bar{P}(\bar{u}_n) + \bar{P}'(\bar{u}_n)(\bar{u}_{n+1} - \bar{u}_n) = 0. \tag{21}$$

In accordance with (19) (or (20)), equation (21) is equivalent to the boundary-value problem

$$\bar{u}_n' - \bar{f}^n + (\bar{u}_{n+1}' - \bar{u}_n') + \bar{f}_{\bar{u}}^n(\bar{u}_{n+1} - \bar{u}_n) = 0, \tag{22}$$

$$\bar{\varphi}^n - \bar{\varphi}_{\bar{u}}^n(\bar{u}_{n+1}(0) - \bar{u}_n(0)) = 0, \tag{23}$$

$$\bar{\psi}^n - \bar{\psi}^n(\bar{u}_{n+1}^0(l) - \bar{u}_n(l)) = 0. \tag{24}$$

Let us verify that condition 1 of Theorem K (see section 1, subsection 1°) is satisfied. By virtue of conditions (α) and (β) of Theorem 2 (which are analogous

to conditions (α) and (β) of Theorem 1), the second Fréchet differential $P''(\bar{u})$ exists and is bounded with respect to the norm in D (for more detail, see the proof of Theorem 1).

Let us verify that condition 2 of Theorem K is satisfied. The existence of the operator $\Lambda_0^{-1}=[P'(\bar{u}_0)]^{-1}$ is equivalent to unique solvability of the linear boundary-value problem (3), (4) for arbitrary right-hand members $\bar{\omega}_0(x)$, $\bar{\omega}_1$, $\bar{\omega}_2$, where $\bar{\omega}_0(x)$ is an arbitrary continuous p-dimensional vector on $[0, l]$ and $\bar{\omega}_1$ and $\bar{\omega}_2$ are arbitrary constant p-dimensional vectors. This solvability will prevail by virtue of the assumption (γ^*). The boundedness of the operator $\Lambda_0^{-1}=[P'(\bar{u}_0)]^{-1}$ which maps the vector $(\bar{\omega}_0(x), \bar{\omega}_1, \bar{\omega}_2) \in Y=C_0[0, l; p]\times R^p \times R^p$ into the vector $\bar{U}(x) \in X$ follows from the representation (12) of the vector $\bar{U}(x)$, which holds by virtue of conditions (γ^*) and (δ^*). Consequently, condition 2 of Theorem K is satisfied. /29

Finally, we shall assume that the number η_0 in inequality (17) is sufficiently small that

$$h_0 = B_0 K \eta_0 < \frac{1}{2}, \tag{25}$$

where $B_0 \geqslant \|\Lambda_0^{-1}\|$ and $K \geqslant \|P''(\bar{u})\|$ in D, and the ball $\mathrm{III}(\bar{u}_0, r_0): \{0 \leqslant x \leqslant l, \|\bar{u}-\bar{u}_0\| \leqslant r_0\}$ is contained in the region $D: \{0 \leqslant x \leqslant l, \|\bar{u}-\bar{u}_0\| < c\}$, if

$$r_0 = N(h_0)\eta_0 = \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0.$$

Then, all conditions of Theorem K are satisfied. This completes the proof of Theorem 2.

Remark: Assuming that the region D is contained in the ball

$$\mathrm{III}(\bar{u}_0, r_1): \{0 \leqslant x \leqslant l, \quad \|\bar{u}-\bar{u}_0\| \leqslant r_1\},$$

where

$$r_1 = \tilde{N}(h_0)\eta_0 = \frac{1+\sqrt{1-2h_0}}{h_0}\eta_0,$$

we see, in accordance with the uniqueness theorem of [1], that a solution in D of the nonlinear problem (1), (2) is unique.

2°. The second variant of Newton's method (the "modified" Newton's method). Starting with the zeroth approximation $\bar{u}_0(x)$ (see equation (1)), we shall seek all subsequent approximations in accordance with the formulas

$$\bar{u}_{n+1} - \bar{f}_{\bar{u}}^0 \bar{u}_{n+1} = \bar{f}^n - \bar{f}_{\bar{u}}^0 \bar{u}_n, \quad 0 \leqslant x \leqslant l, \tag{26}$$

$$\bar{\varphi}_{\bar{u}}^{0}\bar{u}_{n+1}(0) = \bar{\varphi}_{\bar{u}}^{0}\bar{u}_{n}(0) - \bar{\varphi}^{n}, \tag{27}$$

$$\bar{\psi}_{\bar{u}}^{0}\bar{u}_{n+1}(l) = \bar{\psi}_{\bar{u}}^{0}\bar{u}_{n}(l) - \bar{\psi}^{n}, \tag{28}$$

where the notation is as before. We can solve the boundary-value problem (26, 28) for n = 0, 1, 2, ..., for example, by using a sweep [3].

That this variant of Newton's method converges at a rate characterized by the inequality

$$\|\bar{u}_{n} - \bar{u}^{*}\|_{0} \leqslant \frac{\eta_{0}}{h_{0}}(1 - \sqrt{1 - 2h_{0}})^{n+1}, \quad n = 0, 1, \ldots,$$

is proven in a manner analogous to the proof in the case of the first variant of Newton's method [1].

3°. The third variant of Newton's method (with "modified" boundary conditions) is defined by the iterational relations /30

$$\left.\begin{aligned} &\bar{u}'_{n+1} - \bar{f}_{\bar{u}}^{n}\bar{u}_{n+1} = \bar{f}^{n} - \bar{f}_{\bar{u}}^{n}\bar{u}_{n}, \quad 0 \leqslant x \leqslant l, && (29)\\ &\bar{\varphi}_{\bar{u}}^{0}\bar{u}_{n+1}(0) = \bar{\varphi}_{\bar{u}}^{0}\bar{u}_{n}(0) - \bar{\varphi}^{n}, && (30)\\ &\bar{\psi}_{\bar{u}}^{0}\bar{u}_{n+1}(l) = \bar{\psi}_{\bar{u}}^{0}\bar{u}_{n}(l) - \bar{\psi}^{n}, && (31) \end{aligned}\right\} \quad n = 0, 1, 2, \ldots$$

where the notation is as before. We can show though, no longer by a reduction to the theorems in [1], that the successive approximations obtained in accordance with formulas (29) - (30) converge to the exact solution of the problem (1), (2) at a rate characterized by an inequality analogous to inequality (18). However, this requires rather laborious calculations and for this reason we do not present it here [4].

4°. Newton's method in the form of shots. Suppose that we need to solve the boundary-value problem (1), (2). Suppose that

$$\operatorname{rang}\bar{\varphi}_{\bar{u}} = r, \quad \operatorname{rang}\bar{\psi}_{\bar{u}} = s, \quad r + s = p. \tag{32}$$

Then, the values $\bar{u}_1(0), \ldots, \bar{u}_p(0)$ can, in general, be expressed as functions of the s parameters $q_1, q_2, \ldots, q_s$:

$$u_i(0) = \alpha_i(q_1, q_2, \ldots, q_s), \quad i = 1, \ldots, p. \tag{33}$$

Let us write the system of equations (33) more briefly in vector form

$$\bar{u}(0) = \bar{\alpha}(\bar{q}), \tag{34}$$

where $\bar{q} = \{q_1, \ldots, q_s\}$ is an s-dimensional vector. Let us assume that the functions (33) are twice continuously differentiable with respect to $q_1, q_2, \ldots, q_s$ in a convex region in the space $q_1, q_2, \ldots, q_s$. Solving Cauchy's problem

$$\bar{u}' = f(x, \bar{u}),\ 0 \leqslant x \leqslant l, \tag{35}$$

$$\bar{u}(0) = \bar{\alpha}(\bar{q}), \tag{36}$$

we obtain for x = l the values

$$\bar{u}(l) = \bar{\beta}(\bar{q}), \tag{37}$$

or, in scalar form,

$$u_i(l) = \beta_i(q_1, q_2, \ldots, q_s),\ i = 1, 2, \ldots, p. \tag{38}$$

If $\bar{f}$ (x, $\bar{u}$) is a continuous function with respect to all the arguments x, $u_1, \ldots,$ u_p throughout the region D: $\{0 \leqslant x \leqslant l,\ \|\bar{u}\| < c\}$ and has continuous bounded first and second partial derivatives with respect to $u_1, \ldots, u_p$, the functions (38) will be twice continuously differentiable with respect to $q_1, \ldots, q_s$ by virtue of the theorem on differentiability of solutions of a system of differential equations with respect to initial conditions [2]. Let us assume, just as before, that $\psi_{\bar{u}}$ and $\psi_{\bar{u}\bar{u}}$ exist, are continuous, and are bounded in the region in question. Then, /31

$$\bar{\psi} = \bar{\psi}(\bar{\beta}(\bar{q})) = \bar{\psi}[\bar{q}] \tag{39}$$

will be a twice continuously differentiable function with bounded first and second derivatives with respect to $q_1, q_2, \ldots, q_s$ in that region. Obviously, the boundary-value problem (1), (2) will be solved if we succeed in finding a vector $\bar{q}$ such that

$$\bar{\psi}(\bar{q}) = 0. \tag{40}$$

Thus, the problem of solving the boundary-value problem (1), (2) has reduced to solving equation (40). We can apply to equation (40) the usual or modified form of Newton's method. As a result, we arrive at Newton's method or a modified form of it in the form of shots. Let us pause first to look at the basic outline of Newton's method in the form of shots. Suppose that we know the n-th approximation $\bar{q}^n$ of the sought-for vector $\bar{q}$. Then, we find the (n + 1)st approximation in the following manner. Solving Cauchy's problem

$$\bar{u}' = f(x, \bar{u}), \tag{41}$$

$$\bar{u}(0) = \alpha(\bar{q}^n), \tag{42}$$

we obtain

$$\bar{u}(l) = \bar{\beta}(\bar{q}^n),\ \bar{\psi}(\bar{u}(l)) = \bar{\psi}(\bar{\beta}(\bar{q}^n)) = \bar{\psi}[\bar{q}^n] = \bar{\psi}^n. \tag{43}$$

We find $\bar{q}^{n+1}$ from the system of equations

$$\bar{\psi}^n + \bar{\psi}^n_{\bar{q}}(\bar{q}^{n+1} - \bar{q}^n) = 0 \tag{44}$$

For simplicity in our subsequent exposition, let us assume that the first s rows of the matrix $(\partial\psi_i/\partial q_j)$, for i = 1, 2, ..., s, are linearly independent. Then, to solve the system (44), we need as a preliminary to calculate the s^2 partial derivatives

$$\frac{\partial\psi_i}{\partial q_j}\Big|_{\bar{q}=\bar{q}^n}; \quad i=1,2,\ldots,s; \quad j=1,2,\ldots,s. \tag{45}$$

Here,

$$\frac{\partial\psi_i}{\partial q_j}\Big|_{\bar{q}=\bar{q}^n}=\left(\frac{\partial\psi_i}{\partial u_1(l)}\frac{\partial u_1(l)}{\partial q_j}+\ldots+\frac{\partial\psi_i}{\partial u_p(l)}\frac{\partial u_p(l)}{\partial q_j}\right)\Big|_{\bar{q}=\bar{q}^n}, \tag{46}$$

and the partial derivatives $\frac{\partial u_k(l)}{\partial q_j}$, for k = 1, 2, ..., p and j = 1, 2, ..., s, can be found on the basis of the theorem on differentiation of solutions of differential equations with respect to the initial conditions by solving Cauchy's problems for equations in the variations /32

$$\bar{v}'=\bar{f}_{\bar{u}}(x,\bar{u})\bar{v} \tag{47}$$

with suitable initial conditions. We can avoid solving the equations in the variations by finding the partial derivatives (45) in the following manner: We give q_j^n an increment Δq_j and leave the values $q_1^n, \ldots, q_{j-1}^n, q_{j+1}^n, \ldots, q_s^n$ unchanged. Then we solve Cauchy's problem

$$\begin{aligned}\bar{u}'&=f(x,\bar{u}),\ 0\leqslant x\leqslant l,\\ \bar{u}(0)&=\bar{\alpha}(\bar{q}^n+\Delta_j\bar{q}),\end{aligned} \tag{48}$$

where $\Delta\bar{q}=\{0, 0, \ldots, 0, \Delta q_j, 0, \ldots, 0\}$. When we do this, we obtain, for $x = l$, the new values

$$\bar{\psi}^n+\Delta_j\bar{\psi}=\bar{\psi}[\bar{q}^n+\Delta_j\bar{q}] \tag{49}$$

or, in scalar form,

$$\psi_i[q_1^n,\ldots,q_{j-1}^n,q_j^n+\Delta q_j,q_{j+1}^n,\ldots,q_s^n],\ i=1,2,\ldots,s. \tag{50}$$

Then, we set

$$\frac{\partial\psi_i}{\partial q_j}\Big|_{\bar{q}=\bar{q}^n}\approx$$

$$\approx\frac{\psi_i[q_1^n,\ldots,q_{j-1}^n,q_j^n+\Delta q_j,q_{j+1}^n,\ldots,q_s^n]-\psi_i[q_1^n,\ldots,q_{j-1}^n,q_j^n,q_{j+1}^n,\ldots,q_s^n]}{\Delta q_j}. \tag{51}$$

Proceeding in this way for j = 1, 2, ..., s, we obtain all the required partial derivatives.

If we replace the system of equations (44) with the system

$$\bar{\psi}^n + \bar{\psi}_q^0 (\bar{q}^{n+1} - \bar{q}^n) = 0, \tag{52}$$

we obtain the modified scheme of Newton's method in the form of shots.

Application of the basic scheme of Newton's method in the form of shots requires, at every step of the iterations, solution of (s + 1) Cauchy problems for the system of differential equations (41) if the partial derivatives are found from formulas (51) or of a single Cauchy problem for the system (41) and s Cauchy problems for the system of equations in variations (47).

Application of the modified scheme of Newton's method in the form of shots requires only at the first step of the iterations solution of (s + 1) Cauchy problems just in the case of the basic scheme, and at every subsequent step of the iterations it requires solution of only one Cauchy problem for the system (41).

The convergence of Newton's method in the form of shots with suitable estimate of the rate of convergence follows easily from the general theorems on convergence of Newton's method [1] and the theorems on differentiation of solutions of systems of differential equations with respect to intial conditions [2]. /33

Remark: It would not be difficult to write the finite-difference analogues for all variants of Newton's method that were considered in section 2, just as we did in subsection 3° of section 1. Here, the schemes of Newton's method in the form of shots admit direct application of arbitrary finite-difference methods and Runge's method.

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